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Factorizations in Schubert Cells

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For any reduced decomposition $\mathbf{i} = (i_1, i_2, ..., i_N)$ of a permutation w and any ring R we construct a bijection $P_{\mathbf{i}}: (x_1, x_2, ..., x_N) \mapsto P_{i_1}(x_1)\, P_{i_2}(x_2) \cdots P_{i_N}(x_N)$ from R^N to the Schubert cell of w, where $P_{i_1}(x_1), P_{i_2}(x_2), ..., P_{i_N}(x_N)$ stand for certain elementary matrices satisfying Coxeter-type relations. We show how to factor explicitly any element of a Schubert cell into a product of such matrices. We apply this to give a one-to-one correspondence between the reduced decompositions of w and the injective balanced labellings of the diagram of w, and to characterize commutation classes of reduced decompositions. © 2000 Academic Press

Key Words: symmetric group; reduced decomposition; flag variety; Schubert cell; factorization of matrices; commutation class.

Etant donné un anneau R et une décomposition réduite $\mathbf{i} = (i_1, i_2, ..., i_N)$ d'une permutation w, nous construisons une bijection $P_{\mathbf{i}}: (x_1, x_2, ..., x_N) \mapsto P_{i_1}(x_1)$ $P_{i_2}(x_2) \cdots P_{i_N}(x_N)$ de R^N vers la cellule de Schubert de w, où $P_{i_1}(x_1), P_{i_2}(x_2), ...,$ $P_{i_N}(x_N)$ sont des matrices élémentaires vérifiant des relations de type Coxeter. Nous montrons comment factoriser explicitement tout élément d'une cellule de Schubert

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comme un produit de matrices $P_i(x)$. Nous utilisons ces factorisations pour établir une bijection entre les décompositions réduites de w et les remplissages injectifs équilibrés du diagramme de w et pour caractériser les classes de commutation de décompositions réduites.

Mots-Clés: groupe symétrique; décomposition réduite; variété de drapeaux; cellule de Schubert; factorisation de matrices; classe de commutation.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In this article we study matrix products obtained by multiplying certain elementary matrices, investigated in [KR], in the order given by a reduced decomposition of a permutation w. It turns out that such a matrix product differs from the matrix of the permutation w only by the entries lying in the diagram of w. In this way, we get a parametrization of the Schubert cell of w. Such parametrizations were previously known over the complex numbers. Here we extend them to any noncommutative ring together with precise formulas for the entries of the matrix products and determinantal formulas for the inverse mappings. These formulas use planar configurations naturally associated to reduced decompositions. We also show that the linear parts of these parametrizations give exactly all injective balanced labellings of the diagram of w, as defined in [FGRS], and that the quadratic parts characterize the commutation classes of reduced decompositions. Thus, these parametrizations provide a powerful algebraic representation of the diagram of a permutation, an object which comes up in many constructions related to permutations and Schubert varieties. One of the most interesting features of the objects we consider is their triple nature: algebraic, combinatorial, topological. In our proofs we rely in turn on each of these aspects.

Let us state our main results. Given a permutation w of the set $\{1, 2, ..., n\}$, recall from [Mcd, Chap. I] that the *diagram* D_w, first introduced by Rothe in 1800 (cf. [Mu, pp. 59–60]) is the subset of $\{1, ..., n\} \times \{1, ..., n\}$ consisting of all couples $(w(k), j)$ such that $j < k$ and $w(j) > w(k)$, or, equivalently, of all couples (i, j) such that

$$i < w(j) \qquad \text{and} \qquad j < w^{-1}(i). \tag{1.1}$$

It is clear that the cardinality of D_w is equal to the number of inversions of w or, equivalently, to the length $l(w)$ of w with respect to the standard generating set $s_1, ..., s_{n-1}$ of the symmetric group S_n, where s_i is the simple transposition $(i, i+1)$.

Let M_w be the matrix of the permutation w: this is the $n \times n$-matrix defined by

$$(M_w)_{ij} = \delta_{i, w(j)} \tag{1.2}$$

for all $1 \leqslant i, j \leqslant n$. Given a ring R and a permutation $w \in S_n$, we define the subset C_w of the group $GL_n(R)$ of invertible $n \times n$-matrices with entries in R as follows: a matrix is in C_w if it is of the form $M_w + Q$, where Q is any matrix with entries Q_{ij} in R such that $Q_{ij} = 0$ whenever $(i, j) \notin D_w$. It is well known that, when R is a field, the set C_w is in bijection with the Schubert cell BwB/B in the flag variety G/B, where $G = GL_n(R)$ and B is the subgroup of upper triangular matrices (see [Mcd, Appendix, (A.4)]). By extension, we call C_w the *Schubert cell* associated to w.

In [KR] the following invertible $n \times n$-matrices $P_1(x), P_2(x), ..., P_{n-1}(x)$ depending on a variable x were defined: for $1 \leqslant i \leqslant n-1$ the matrix $P_i(x)$ is obtained from the identity matrix by inserting the block

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$$

on the diagonal such that x is the (i, i)-entry of $P_i(x)$. In other words, $P_i(x)$ is of the form

$$\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & & & & & & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & x & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & & & & & & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}. \tag{1.3}$$

As observed in [KR, Lemma 1], the matrices $P_i(x)$ $(1 \leqslant i \leqslant n-1)$ satisfy the following Coxeter-type relations, where x, y, z are elements in a ring R,

$$P_i(x)\, P_j(y) = P_j(y)\, P_i(x) \tag{1.4a}$$

if $|i-j| \geqslant 2$, and

$$P_i(x)\, P_{i+1}(y)\, P_i(z) = P_{i+1}(z)\, P_i(y+xz)\, P_{i+1}(x). \tag{1.4b}$$

Relation (1.4b) is similar to but different from the famous Yang–Baxter equation with spectral parameters

$$P_i(x)\, P_{i+1}(x+z)\, P_i(z) = P_{i+1}(z)\, P_i(x+z)\, P_{i+1}(x),$$

which is another deformation of the usual Coxeter relation. It is interesting to note that the Yang–Baxter equation also leads to properties of Schubert varieties (see, e.g., [LLT]).

Before we state the first result of the paper, let us recall that a sequence $(i_1, i_2, ..., i_N)$ of indices belonging to $\{1, ..., n-1\}$ is a *reduced decomposition* of $w \in S_n$ if $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ and if its length N is equal to the number of inversions of w.

1.1. THEOREM. *For any ring R and any reduced decomposition $\mathbf{i} = (i_1, i_2, ..., i_N)$ of the permutation $w \in S_n$, the map*

$$P_{\mathbf{i}}\colon (x_1, x_2, ..., x_N) \mapsto P_{i_1}(x_1)\, P_{i_2}(x_2) \cdots P_{i_N}(x_N)$$

is a bijection from R^N to the Schubert cell $C_w \subset GL_n(R)$.

Theorem 1.1 is known when R is a field (cf. [Sp, Lemma 10.2.6]; see also [FZ, Proposition 2.11]). The proof we give for an arbitrary ring in Subsection 2.5 is based on Relations (1.4a), (1.4b) and on a matrix identity (given in Proposition 2.1), which is of independent interest.

Let us illustrate this theorem with the permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix} \in S_5. \tag{1.5}$$

It is of length 9 and the Schubert cell C_w consists of all 5×5-matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 1 \\ a_{21} & a_{22} & a_{23} & 1 & 0 \\ a_{31} & a_{32} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.6}$$

The sequence $\mathbf{i} = (1, 2, 3, 4, 1, 2, 3, 1, 2)$ is a reduced decomposition of w. The matrix $P_{\mathbf{i}}(x_1, x_2, ..., x_9)$ is of the form (1.6). Computing its entries a_{ij} in terms of the variables $x_1, x_2, ..., x_9$, we get

$$\begin{gathered} a_{11} = x_3 + x_2 x_8 + x_1 x_6 + x_1 x_5 x_8, \qquad a_{12} = x_4 + x_2 x_9 + x_1 x_7 + x_1 x_5 x_9, \\ a_{13} = x_2 + x_1 x_5, \qquad a_{14} = x_1, \\ a_{21} = x_6 + x_5 x_8, \qquad a_{22} = x_7 + x_5 x_9, \\ a_{23} = x_5, \qquad x_{31} = x_8, \qquad a_{32} = x_9. \end{gathered} \tag{1.7}$$

We see that each entry of $P_{\mathbf{i}}(x_1, x_2, ..., x_9)$ lying in the diagram of w is the sum of some variable x_i and possibly of distinct monomials in the remaining variables (thus, it is a substraction-free polynomial). Moreover, any monomial in the variables x_i appears at most once in the matrix.

These two features hold for $P_{\mathbf{i}}(x_1, x_2, ..., x_N)$, where $\mathbf{i}$ is any reduced decomposition. They follow from Proposition 3.2, which exhibits a general formula for the entries of the matrix $P_{\mathbf{i}}(x_1, ..., x_N)$ in terms of the variables $x_1, ..., x_N$.

In Theorem 3.3 we shall give a formula for the inverse map $P_{\mathbf{i}}^{-1}\colon C_w \to R^N$. This formula yields a factorization of any element of the Schubert cell C_w as a product of matrices $P_i(x)$. There is such a factorization for any reduced decomposition of w. As an application of Theorem 3.3, the matrix (1.6) is the product of the nine factors

$$P_1(a_{14})\, P_2\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & 1 \end{vmatrix}\right) P_3\left(-\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & 1 \\ a_{31} & 1 & 0 \end{vmatrix}\right) P_4\left(-\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & 1 \\ a_{32} & 1 & 0 \end{vmatrix}\right)$$

$$\times P_1(a_{23})\, P_2\left(\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & 1 \end{vmatrix}\right) P_3\left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & 1 \end{vmatrix}\right) P_1(a_{31})\, P_2(a_{32}). \tag{1.8}$$

The expressions appearing in the factorization (1.8) are, up to sign, minors of the matrix (1.6). We shall give a sense to these minors over noncommutative rings as well.

The explicit formulas for the maps $P_{\mathbf{i}}$ and $P_{\mathbf{i}}^{-1}$ will be expressed with the help of the so-called pseudo-line arrangement of the reduced decomposition $\mathbf{i}$. This is a configuration of lines in a horizontal strip of the plane whose precise definition is recalled in Subsection 3.1.

Out of the formulas for $P_{\mathbf{i}}$ and $P_{\mathbf{i}}^{-1}$ it is possible to relate the parametrizations corresponding to two different reduced decompositions (see also Remark 6.2).

Fomin, Greene, Reiner, and Shimozono [FGRS] constructed a one-to-one correspondence between the reduced decompositions of a permutation w and what they call "injective balanced labellings" of the diagram of w. Theorem 1.2 below states that such a bijection can be recovered from the linear part of our parametrization.

Let $Q_{\mathbf{i}}(x_1, ..., x_N)$ be the linear part of the matrix $P_{\mathbf{i}}(x_1, ..., x_N)$, i.e., the matrix obtained from $P_{\mathbf{i}}(x_1, ..., x_N)$ by removing all monomials in $x_1, ..., x_N$ of degree $\neq 1$. As $P_{\mathbf{i}}(x_1, ..., x_N) - M_w$ is entirely supported by the diagram D_w, so is $Q_{\mathbf{i}}(x_1, ..., x_N)$. The matrix $Q_{\mathbf{i}}(x_1, ..., x_N)$ provides each element of the diagram (which we view as a matrix) with a label x_i. Let us denote this labelling by $L_{\mathbf{i}}$. For simplicity, we replace each variable x_i in $L_{\mathbf{i}}$ by its index i.

Such a labelling is *injective* in the terminology of [FGRS], i.e., each integer 1, ..., N appears exactly once as a label in $L_{\mathbf{i}}$.

For the reduced decomposition $\mathbf{i}=(1, 2, 3, 4, 1, 2, 3, 1, 2)$ considered above, (1.7) implies that

$$Q_{\mathbf{i}}(x_1,..., x_9)=\begin{pmatrix} x_3 & x_4 & x_2 & x_1 & 0 \\ x_6 & x_7 & x_5 & 0 & 0 \\ x_8 & x_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.9}$$

which leads to the injective labelling (we indicate the nonzero entries of M_w with crosses)

$$L_{\mathbf{i}}=\begin{pmatrix} 3 & 4 & 2 & 1 & \times \\ 6 & 7 & 5 & \times & \cdot \\ 8 & 9 & \times & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \times & \cdot & \cdot & \cdot \end{pmatrix}. \tag{1.10}$$

The concept of balanced labelling of the diagram of a permutation we use (see Section 4 for the definition) is different from, but equivalent, up to transposition, to the one in [FGRS]. Under this equivalence, the bijection stated in the following theorem is the same as the bijection in [FGRS].

1.2. THEOREM. *For any permutation $w \in S_n$, the map $\mathbf{i} \mapsto L_{\mathbf{i}}$ is a bijection between the set of reduced decompositions of w and the set of injective balanced labellings of the diagram of w.*

Note that this implies that the matrix $P_{\mathbf{i}}(x_1, ..., x_N)$ is completely determined by its linear part (see (4.2) for an explicit formula).

We now use the quadratic part of $P_{\mathbf{i}}(x_1, ..., x_N)$ to get additional information on the set $\mathscr{R}(w)$ of reduced decompositions of w. More precisely, let $R_{\mathbf{i}}$ be the matrix (whose entries are nonnegative integers) which is the coefficient of X^2 in the expansion of $P_{\mathbf{i}}(X, X, ..., X)$ as a polynomial in X. We have

$$P_{\mathbf{i}}(X, X, ..., X) \equiv M_w + Q_{\mathbf{i}}(1, 1, ..., 1)\, X + R_{\mathbf{i}} X^2 \qquad \text{modulo } X^3. \tag{1.11}$$

An equivalent definition will be given in Lemma 4.9. The matrix $R_{\mathbf{i}}$ is a (noninjective) labelling of D_w. We use this matrix to characterize the commutation classes in $\mathscr{R}(w)$. Let us recall what these are.

By a well-known theorem of Iwahori and Tits (see [Bo]), any two reduced decompositions of w can be obtained from each other via a finite sequence of 2-moves and 3-moves. A *2-move* on a reduced decomposition $\mathbf{i}$ consists in replacing two adjacent indices i, j in $\mathbf{i}$ by j, i under the condition $|i-j|>1$. A *3-move* on $\mathbf{i}$ consists in replacing three adjacent indices i, j, i in $\mathbf{i}$ by j, i, j under the condition $|i-j|=1$. Two reduced decompositions $\mathbf{i}$ and $\mathbf{j}$ are said to belong to the same *commutation class* if they can be obtained from each other via a sequence of 2-moves. Commutation classes have recently come up in relation with dual canonical bases for quantum groups (see [BZ, LZ]).

1.3. THEOREM. *Two reduced decompositions* $\mathbf{i}$ *and* $\mathbf{j}$ *of a permutation* w *belong to the same commutation class if and only if* $R_{\mathbf{i}}=R_{\mathbf{j}}$.

The paper is organized as follows. In Section 2 we establish a matrix identity from which we derive Theorem 1.1. In Section 3 we produce explicit formulas for the parametrization $P_{\mathbf{i}}$ (Proposition 3.2) and for the inverse map $P_{\mathbf{i}}^{-1}$ (Theorem 3.3). We deal with balanced labellings and prove Theorem 1.2 in Section 4. Section 5 is devoted to the proof of Theorem 1.3; we also give a characterization of the labellings $R_{\mathbf{i}}$. In Section 6 we consider a partial order on the set $\mathscr{C}(w)$ of commutation classes of reduced decompositions of w, due to Manin and Schechtman; the poset $\mathscr{C}(w)$ has a unique minimal element and a unique maximal element, which we describe explicitly.

We thank Robert Bédard for pointing out Lemma 10.2.6 of [Sp] and for showing us the experimental data he collected on the posets $\mathscr{C}(w_0)$ of Section 6.

2. A MATRIX IDENTITY

Let w be a permutation of the set $\{1, ..., n\}$ and M_w be its permutation matrix. We complete M_w with stars at all places $(w(k), j)$ such that $j<k$ and $w(j)>w(k)$. As we saw in the Introduction, the pattern formed by these stars is the diagram D_w.

Replace the stars by elements $a_1, ..., a_N$ of a ring R in the following order: start with the first column from bottom to top, then the second one again from bottom to top, and so on. We get a matrix $M_w(a_1, ..., a_N)$ whose entries are either 1, 0, or $a_1, ..., a_N$. By definition, the Schubert cell C_w is the set of all matrices $M_w(a_1, ..., a_N)$ when $a_1, ..., a_N$ run over R. Observe that $M_w(0, ..., 0)=M_w$.

We now create a sequence $(j_1, ..., j_N)$ of integers by labelling the entries in the diagram of w (considered as an $n\times n$-matrix) in the following way: the upper entry of the jth column gets the label j; the entry immediately below gets the label $j+1$, and so on. Then $(j_1, ..., j_N)$ is the sequence obtained by reading the labels from bottom to top in each column, one column after the other, starting with the first column. We call $(j_1, ..., j_N)$ the *canonical sequence* of w.

We now express the matrix $M_w(a_1, ..., a_N)$ as a product of N matrices of type $P_i(x)$, as defined in the Introduction.

2.1. PROPOSITION. *Let w be a permutation and $(j_1, ..., j_N)$ be its canonical sequence. Then*

$$M_w(a_1, ..., a_N) = P_{j_1}(a_1)\, P_{j_2}(a_2) \cdots P_{j_N}(a_N).$$

The proof is postponed to Subsection 2.6.

2.2. EXAMPLE. In order to get the canonical sequence for the permutation (1.5), we label its diagram as explained above. We get the labelling

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdot \\ 2 & 3 & 4 & \cdot & \cdot \\ 3 & 4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

from which we see that the canonical sequence is $(3, 2, 1, 4, 3, 2, 4, 3, 4)$. Applying Proposition 2.1, we have

$$\begin{pmatrix} a_3 & a_6 & a_8 & a_9 & 1 \\ a_2 & a_5 & a_7 & 1 & 0 \\ a_1 & a_4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$
$$= P_3(a_1)\, P_2(a_2)\, P_1(a_3)\, P_4(a_4)\, P_3(a_5)\, P_2(a_6)\, P_4(a_7)\, P_3(a_8)\, P_4(a_9).$$

Let us derive a few consequences from Proposition 2.1. The following corollary has been observed in [La, p. 305] (see also [Pa]).

2.3. COROLLARY. *The canonical sequence $(j_1, j_2, ..., j_N)$ of the permutation w is a reduced decomposition of w.*

Proof. Set $a_1 = \cdots = a_N = 0$ in the matrix identity of Proposition 2.1. On the left-hand side, we get the permutation matrix M_w. As for the right-hand side, observe that $P_i(0)$ is the matrix of the transposition s_i. The matrix identity thus implies that $w = s_{i_1} s_{i_2} \cdots s_{i_N}$. The decomposition of w

we get in this way is reduced because its length is equal to the number of inversions of w. ∎

2.4. Corollary. *Let K be a field and let $a_1, ..., a_N$ be variables. If $M_w(a_1, ..., a_N) = P_{i_1}(f_1) \cdots P_{i_k}(f_k)$ where $f_1, ..., f_k$ belong to an extension of the field $K(a_1, ..., a_N)$, then $k \geqslant N$. If, in addition, $k = N$, then $(i_1, ..., i_N)$ is a reduced decomposition of w.*

Proof. If $k < N$, then the degree of transcendence of $a_1, ..., a_N$ would be $< N$, which is not possible. Assume now that $k = N$. Then $f_1, ..., f_k$ are algebraically independent over K and $a_1, ..., a_N$ belong to the ring generated by $f_1, ..., f_k$. Thus we may set $f_i = 0$ for all $i = 1, ..., N$ and we get $M_w(\bar{a}_1, ..., \bar{a}_N) = P_{i_1}(0) \cdots P_{i_k}(0)$, where $\bar{a}_1, ..., \bar{a}_N$ belong to the ground field K. The right-hand side of the previous identity is a permutation matrix $M_{w'}$ with n nonzero entries, whereas the left-hand side is a matrix of the form $M_w + Q$, where all entries of Q not in the diagram of w vanish. In particular, M_w and Q have disjoint supports. Counting the number of nonzero entries in $M_w + Q$, we see that necessarily $Q = 0$ and $M_w = M_{w'}$. It follows that $(i_1, ..., i_N)$ is a reduced decomposition of w. ∎

2.5. *Proof of Theorem* 1.1. By Proposition 2.1, the theorem holds for the canonical reduced decomposition $(j_1, ..., j_N)$. Now one passes from $(j_1, ..., j_N)$ to $(i_1, ..., i_N)$ by a sequence of 2-moves and 3-moves. Relations (1.4a), (1.4b) show how the matrices $P_i(x)$ are affected under such moves. We conclude immediately that

$$M_w(a_1, ..., a_N) = P_{j_1}(a_1)\, P_{j_2}(a_2) \cdots P_{j_N}(a_N) = P_{i_1}(a'_1)\, P_{i_2}(a'_2) \cdots P_{i_N}(a'_N),$$

where $a_k \mapsto a'_k$ is an endomorphism of the free ring generated by $a_1, ..., a_N$. This endomorphism is a product of Jonquières automorphisms, hence it is an automorphism. Recall that a Jonquières automorphism is an automorphism that sends some variable to the sum of itself and of a polynomial in the other variables, and fixes the remaining variables (see [Co, p. 342]). We may then conclude that the matrices $P_{i_1}(a_1)\, P_{i_2}(a_2) \cdots P_{i_N}(a_N)$ parametrize the Schubert cell C_w as well. ∎

2.6. *Proof of Proposition* 2.1. We proceed by induction on the length N of w. Suppose that $N = 1$. Then w is a transposition s_j. The only element of D_{s_j} being (j, j), the canonical sequence of w is (j) and the identity $M_w(a_1) = P_j(a_1)$ holds trivially.

Suppose now that $N > 1$ and that we have proved the proposition for all permutations $w' \in S_n$ of length $< N$. Let (i, j) be the position of the entry a_N in the matrix $M_w(a_1, ..., a_N)$.

a

$$\begin{array}{c c}
 & \begin{array}{cccc} \quad & j & j+1 & \quad \end{array} \\
i & \left(\begin{array}{cccc} & a_N & 1 & \\ & & & \\ & & & \\ & & & \\ & 1 & & \end{array}\right)
\end{array}$$

b

$$\begin{array}{c}
\begin{array}{cccc} \quad & j & j+1 & \quad \end{array} \\
\left(\begin{array}{cccc} & j & 1 & \\ & j+1 & & \\ & j+2 & & \\ & \cdots & & \\ & 1 & & \end{array}\right)
\end{array}$$

c

$$\begin{array}{c}
\begin{array}{cccc} \quad & j & j+1 & \quad \end{array} \\
\left(\begin{array}{cccc} & 1 & & \\ & & j+1 & \\ & & j+2 & \\ & & \cdots & \\ & & 1 & \end{array}\right)
\end{array}$$

FIG. 2.1. (a) $M_w(a_1, ..., a_N)$; (b) canonical sequence of w; (c) canonical sequence of w'.

Claim 1. $j<n$ and $w(j+1)=i$. This follows from the definitions of D_w and of the canonical sequence. See Fig. 2.1a.

Claim 2. If $w'=ws_j$, then $D_{w'}$ is obtained from D_w as follows: first remove the element $(i, j) \in D_w$, then exchange the jth, and the $j+1$st columns. This claim is easy to prove (see Figs. 2.1a–2.1c). As a consequence, $D_{w'}$ has one element less than D_w, and the length of w' is $N-1$.

Claim 3. By definition of the canonical sequence $(j_1, ..., j_N)$ of w, we have $j_N=j$, $j_{N-1}=j+1$, $j_{N-2}=j+2$, and so forth, as long as we are in the jth column of D_w. Let $w'=ws_j$ as in Claim 2 and let $(j'_1, ..., j'_{N-1})$ be its canonical sequence. Then $j'_r=j_r$ for all $r=1, ..., N-1$. The proof of this claim uses Claim 2 and the fact that the columns lying left to the jth column are the same in D_w and in $D_{w'}$ (see Figs. 2.1b and 2.1c).

Claim 4. Let $w'=ws_j$ be as in Claim 2. The matrix $M_{w'}(a_1, ..., a_{N-1})$ is obtained from $M_w(a_1, ..., a_N)$ by first replacing a_N by 0, then by exchanging the jth and the $j+1$st columns. This follows from Claim 2.

Claim 5. Given any matrix M, the $j+1$st column of the matrix product $MP_j(a_N)$ is equal to the jth column of M, and the jth column of $MP_j(a_N)$ is the sum of the $j+1$st column of M and of the jth column of M multiplied by a_N; all other columns of $MP_j(a_N)$ and M are the same.

Claims 3–5 and the induction hypothesis imply that

$$\begin{aligned}
M_w(a_1, ..., a_N) &= M_{w'}(a_1, ..., a_{N-1})\, P_j(a_N) = P_{j'_1}(a_1) \cdots P_{j'_{N-1}}(a_{N-1})\, P_j(a_N) \\
&= P_{j_1}(a_1) \cdots P_{j_{N-1}}(a_{N-1})\, P_{j_N}(a_N). \quad \blacksquare
\end{aligned}$$

3. FACTORIZATIONS IN A SCHUBERT CELL

The aim of this section is to give explicit formulas for the parametrization $P_{\mathbf{i}}$ and for the inverse maps $P_{\mathbf{i}}^{-1}$.

3.1. *Pseudo-Line Arrangements.* Following [BFZ, Sect. 2.3] we define a *pseudo-line arrangement* as the union of a finite number of intervals, called *pseudo-lines*, smoothly immersed in a bounded horizontal strip of the plane such that

(i) each vertical line in the strip intersects each pseudo-line in exactly one point,

(ii) each pair of distinct pseudo-lines intersects at one point at most, such a intersection being transversal (we call the intersection of two pseudo-lines a *crossing point*),

(iii) no three pseudo-lines meet at a point and no two crossing points lie on the same vertical line. (See Figs. 3.2 and 3.3 for two examples of pseudo-line arrangements.)

We consider pseudo-line arrangements up to isotopy in the space of pseudo-line arrangements.

A *path* on a pseudo-line arrangement is a subset that projects bijectively onto the horizontal projection of the pseudo-line arrangement. Thus a path is composed of parts of one or more pseudo-lines. A path is *admissible* if the only allowed changes of pseudo-lines are at crossing points where the left tangent of the path has a bigger slope than the right tangent (see Fig. 3.1 for an allowed change of pseudo-lines).

Let us label the left and right ends of the pseudo-lines in a pseudo-line arrangement by the integers 1 to n from bottom to top, where n is the number of pseudo-lines. Any reduced decomposition $(i_1, ..., i_N)$ of a permutation $w \in S_n$ gives rise to a pseudo-line arrangement with n pseudo-lines, according to the following rules:

(a) to the transposition s_i we assign the pseudo-line arrangement with a unique crossing point, which sits on the pseudo-lines whose left (and right) labels are i and $i+1$ (see Fig. 3.2);

(b) if D' is the pseudo-line arrangement of $(i_1, ..., i_{N-1})$ and D'' is the pseudo-line arrangement of (i_N), then the pseudo-line arrangement we assign to $(i_1, ..., i_N)$ is obtained by placing D'' to thy right of D' in the horizontal strip and gluing together the ith left end of D'' and the ith right end of D' for all $i = 1, ..., n$.

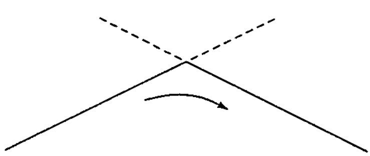


FIG. 3.1. Allowed change of pseudo-lines in an admissible path.

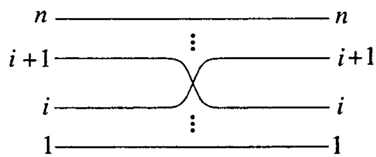


FIG. 3.2. Pseudo-line arrangement of the transposition s_i.

The pseudo-line arrangement of a reduced decomposition has the following properties:

(i) If the right label of a pseudo-line is k, then its left label is $w(k)$.

(ii) The number N of crossing points is equal to the length of the reduced decomposition.

Given a reduced decomposition $\mathbf{i}$ of length N and its pseudo-line arrangement Γ, we label the N crossing points of Γ from left to right by labels $x_1, ..., x_N$ considered in this order. We define the *weight* of a path in Γ as the product from left to right of the labels of the crossing points where the path switches from a pseudo-line to another. If a path consists of one pseudo-line (without any pseudo-line change), we agree that its weight is 1.

We use these definitions to describe the entries of the matrix $P_{\mathbf{i}}(x_1, ..., x_N)$.

3.2. PROPOSITION. *For all* $1 \leqslant i, j \leqslant n$, *the* (i, j)*-entry of the matrix* $P_{\mathbf{i}}(x_1, x_2, ..., x_N) = P_{i_1}(x_1) \cdots P_{i_N}(x_N)$ *is equal to the sum of the weights of all admissible paths with left label* i *and right label* j *in the pseudo-line arrangement of* $\mathbf{i}$.

Proof. If $\mathbf{i} = (i)$ is of length one, then we immediately see from Fig. 3.2 that Proposition 3.2 holds for the entries of $P_{\mathbf{i}}(x_1) = P_i(x_1)$. The case of a reduced decomposition $\mathbf{i}$ of length >1 follows by induction on the length of $\mathbf{i}$ from the previous case and from the formula giving the entries of a product of matrices. ∎

Apply Proposition 3.2 to Fig. 3.3 in order to recover Formulas (1.7).

Our next task is to give a formula for the inverse map $P_{\mathbf{i}}^{-1}: C_w \to R^N$, where $\mathbf{i}$ is a reduced decomposition of w. To the kth crossing point (counted from left to right) in the pseudo-line arrangement of $\mathbf{i}$ we associate subsets I_k, J_k of $\{1, ..., n\}$ and a sign ε_k as follows. Considering the two pseudo-lines intersecting at this point, we define by T_k the union of their parts lying below the crossing point. We call T_k the kth *roof* of the pseudo-line arrangement. A roof has a left and a right slope. A pseudo-line

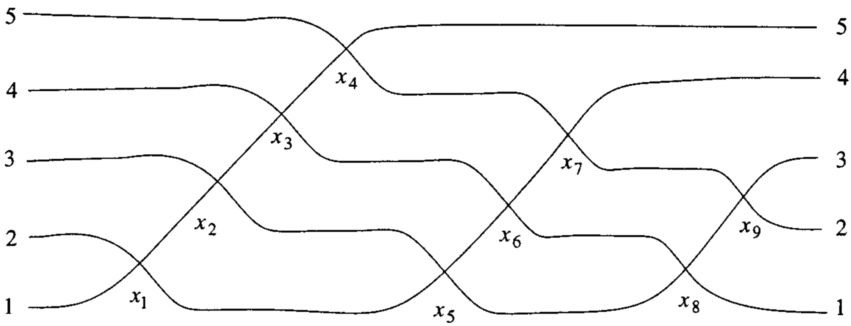


FIG. 3.3. Pseudo-line arrangement of the reduced decomposition (1, 2, 3, 4, 1, 2, 3, 1, 2).

will be called *efficient* for T_k if it intersects transversally both the left and the right slopes of T_k.

We set $\varepsilon_k = +$ (resp. $\varepsilon_k = -$) if the total number of pairwise intersections of the efficient pseudo-lines for T_k is even (resp. is odd). In other words, ε_k is the sign of the permutation represented by the pseudo-line arrangement consisting only of the pseudolines that are efficient for T_k. The set I_k (resp J_k) is defined as the set of left labels (resp. right labels) of the efficient pseudo-lines for T_k to which we add the left label (resp. right label) of the left slope (resp. right slope) of T_k. (See Remark 3.10 for an alternative definition of I_k and J_k.)

In order to state Theorem 3.3, we need the following two conventions. Given a $n \times n$-matrix M and subsets I, J of $\{1, ..., n\}$, we denote by $M_{I,J}$ the submatrix of M consisting of the entries M_{ij} of M such that $i \in I$ and $j \in J$. For a $p \times p$-matrix $A = (A_{ij})_{1 \leqslant i, j \leqslant p}$, we set

$$|A| = \sum_{\sigma \in S_p} (-1)^{\sigma} A_{1, \sigma(1)} A_{2, \sigma(2)} \cdots A_{p, \sigma(p)}. \tag{3.1}$$

This coincides with the determinant of A when the ring R is commutative.

We now give a formula for the inverse bijection $P_{\mathbf{i}}^{-1}\colon C_w \to R^N$.

3.3. Theorem. *Let $\mathbf{i} = (i_1, i_2, ..., i_N)$ be a reduced decomposition of a permutation $w \in S_n$ and let M be an element of the Schubert cell C_w. Then for any k such that $1 \leqslant k \leqslant N$, the kth component of $P_{\mathbf{i}}^{-1}(M) = (x_1, ..., x_N) \in R^N$ is given by*

$$x_k = \varepsilon_k \, |M_{I_k, J_k}|. \tag{3.2}$$

For example, consider the roof T_4 (labelled x_4) in Fig. 3.3. The left label of its left slope is 1 and the right label of its right slope is 2. The roof T_4

has two efficient pseudo-lines: one connects 2 to 4, the other connects 3 to 3, from left to right. It follows that the index sets are $I_4 = \{1, 2, 3\}$ and $J_4 = \{2, 3, 4\}$. We have $\varepsilon_4 = -$ because the two efficient pseudo-lines intersect exactly once. Proceeding in the same way for the other roofs, we recover the factorization (1.8).

3.4. *Proof of Theorem* 3.3: *Preliminaries.* We shall prove Theorem 3.3 in two steps: first, in the case when the ring R is commutative (see Subsection 3.6); then in the case of a general ring R (see Subsection 3.9).

In view of Theorem 1.1, it is enough to check (3.2) when $M = P_{\mathbf{i}}(x_1, ..., x_N)$. We start with the following construction.

Let us concentrate on the kth crossing point (counted from left to right) in the pseudo-line arrangement Γ of $\mathbf{i}$. This crossing point is labelled by x_k. Out of Γ we form a new pseudo-line arrangement Γ' by keeping only the two pseudo-lines intersecting at the kth crossing point of Γ and the pseudo-lines that are efficient for the kth roof T_k. The pseudo-line arrangement Γ' has m pseudo-lines with $m \leqslant n$. The labels of the crossing points of Γ' form a subset $\{y_1, ..., y_M\}$ of $\{x_1, ..., x_N\}$ (we assume that the labels $y_1, ..., y_M$ appear in this order when we sweep Γ' from left to right). Observe that the label x_k belongs to this subset: we denote by p the integer such that $x_k = y_p$. The configuration Γ' is the pseudo-line arrangement of a reduced decomposition $\mathbf{j}$. The corresponding permutation in S_m will be denoted by w'.

From the reduced decomposition $\mathbf{j} = (j_1, ..., j_M)$ and the labels $y_1, ..., y_M$ of the crossing points of Γ', we can form the matrix product $P_{j_1}(y_1) \cdots P_{j_M}(y_M) \in GL_m(R)$. Removing the last row and the last column in each matrix $P_{j_l}(y_l)$ appearing in the previous product, we get $(m-1) \times (m-1)$-matrices $\bar{P}_{j_l}(y_l)$. Let us form the $(m-1) \times (m-1)$-matrix

$$M_k = \bar{P}_{j_1}(y_1) \cdots \bar{P}_{j_M}(y_M). \tag{3.3}$$

We claim the following, which relates the matrix M_k to the submatrix M_{I_k, J_k} of $M = P_{\mathbf{i}}(x_1, ..., x_N)$, as defined above.

3.5. Lemma. *For all $k = 1, ..., N$, we have $M_k = M_{I_k, J_k}$.*

Proof. As a consequence of the definitions of I_k, J_k, and of Proposition 3.2, the submatrix M_{I_k, J_k} is the $(m-1) \times (m-1)$-matrix obtained from the $m \times m$-matrix $P_{j_1}(y_1) \cdots P_{j_M}(y_M)$ by removing its last row and its last column, in view of the following observation: if l is a pseudo-line that is efficient for the roof T_h formed by two pseudo-lines l_1, l_2 that are efficient for the roof T_k, then l is efficient for T_k. Figure 3.4 gives graphical evidence for this observation.

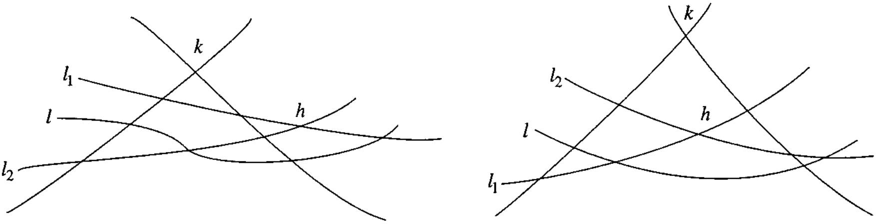


FIGURE 3.4

It therefore suffices to check that the operation of simultaneously removing the last row and the last column of a matrix commutes with the product of matrices. This is of course not true in general. Nevertheless, it works for the product $P_{j_1}(y_1) \cdots P_{j_M}(y_M)$.

Indeed, there is exactly one integer p such that $y_p = x_k$. It is clear from the definition of Γ' that we have $j_p = m-1$ for the corresponding index j_p. Hence $P_{j_p}(y_p) = P_{m-1}(x_k)$ is of the form

$$\begin{pmatrix} & & & & 0 \\ & \bar{P}_{j_p}(y_p) & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

where the entries of the last row and of the last column are all 0, except the $(m, m-1)$- and the $(m-1, m)$-entries, which are equal to 1.

By contrast, if $l \neq p$, then $j_l < m-1$, which implies that $P_{j_l}(y_l)$ is of the form

$$\begin{pmatrix} & & & & 0 \\ & \bar{P}_{j_l}(y_l) & & & \vdots \\ & & & & 0 \\ & & & & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

where now the only nonzero entry in the last row and in the last column is the (m, m)-entry.

In view of these facts, the equality $M_k = M_{I_k, J_k}$ will then be a consequence of the following matrix identity holding for any triple (A, B, C) of $(m-1) \times (m-1)$-matrices,

$$\begin{pmatrix} & A & & 0\\ & & & \vdots\\ & & & 0\\ & & & 0\\ 0 & \cdots & 0\ \ 0 & 1 \end{pmatrix}\begin{pmatrix} & B & & 0\\ & & & \vdots\\ & & & 0\\ & & & 1\\ 0 & \cdots & 0\ \ 1 & 0 \end{pmatrix}\begin{pmatrix} & C & & 0\\ & & & \vdots\\ & & & 0\\ & & & 0\\ 0 & \cdots & 0\ \ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & ABC & & *\\ & & & \vdots\\ & & & *\\ & & & *\\ * & \cdots & *\ \ * & 0 \end{pmatrix},$$

where ABC is the product of the matrices A, B, C. ∎

3.6. *Proof of Theorem* 3.3: *The Commutative Case.* Assume that the ring R is commutative. Then we may take the determinants of both sides of (3.3) and we get

$$|M_k| = |\bar{P}_{j_1}(y_1)| \cdots |\bar{P}_{j_M}(y_M)|. \tag{3.4}$$

We saw in the proof of Lemma 3.5 that $P_{j_p}(y_p) = P_{m-1}(x_k) \in GL_m(R)$. Consequently, $\bar{P}_{j_p}(y_p)$ is a diagonal matrix whose diagonal entries are 1, except for the last one which is $y_p = x_k$. It follows that $|\bar{P}_{j_p}(y_p)| = x_k$. If $l \neq p$, then $|\bar{P}_{j_l}(y_l)| = |P_{j_l}(y_l)| = -1$. The computation of these determinants, together with (3.4) and Lemma 3.5, implies that $|M_{I_k, J_k}| = \varepsilon x_k$ for some $\varepsilon = \pm$.

In order to complete the proof of (3.2), it remains to check that $\varepsilon = \varepsilon_k$. Each factor $|\bar{P}_{j_l}(y_l)| = |P_{j_l}(y_l)|$ with $p \neq l$ corresponds to a crossing point of Γ' that is not the top of the roof T_k in Γ. Therefore, ε is the parity of the number of such crossing points. The latter can be divided into two sets: the crossing points that lie on the slopes of the roof T_k and the crossing points that do not. By the very definition of efficient pseudo-lines, the crossing points that lie on T_k come in pairs. It follows that ε is the parity of the number of the crossing points of Γ' that do not lie on T_k; now, the latter are exactly the intersection points of pairs of efficient pseudo-lines. This proves that $\varepsilon = \varepsilon_k$. ∎

For the noncommutative case, we shall need the following lemma, which is of independent combinatorial interest. In order to state it, we shall need the concept of a *traverse* of a matrix A: this is any nonzero monomial $A_{1,\sigma(1)} A_{2,\sigma(2)} \cdots A_{p,\sigma(p)}$ occurring in the right-hand side of (3.1).

$$\begin{pmatrix}
* & * & * & * & * & * & * & \underline{*} & \overline{*} & 1 \\
* & * & * & \overline{\underline{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & * & \underline{*} & * & \overline{1} & 0 & 0 \\
* & * & * & 0 & * & * & \overline{*} & * & \underline{1} & 0 \\
* & * & \overline{\underline{1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & \underline{*} & \overline{1} & 0 & 0 & 0 & 0 \\
\underline{*} & * & 0 & 0 & \overline{1} & 0 & 0 & 0 & 0 & 0 \\
* & \overline{\underline{1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{*} & 0 & 0 & 0 & 0 & 0 & \underline{1} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

FIGURE 3.5

3.7. LEMMA. *Let $\sigma \in S_m$ such that $\sigma(1) = m$ and $\sigma(m) = 1$, and Q be any $m \times m$-matrix whose support lies in the diagram of σ. Let A be the matrix obtained from $M_\sigma + Q$ by deleting its last row and its last column. Then any traverse of A is of the form*

$$A_{1, j_1} A_{\sigma(j_1), j_2} A_{\sigma(j_2), j_3} \cdots A_{\sigma(j_{p-1}), j_p}$$

with $j_1 > j_2 > \cdots > j_{p-1} > j_p = 1$.

Lemma 3.7 is illustrated in Fig. 3.5: overlining and underlining certain entries, we show two examples of traverses of the matrix obtained by removing the last row and the last column.

Proof. The lemma clearly holds if $m = 1$. Assume $m > 1$ and the lemma holds for any $m' < m$ and any $\sigma' \in S_{m'}$ such that $\sigma(1) = m'$ and $\sigma(m') = 1$.

Consider $\sigma \in S_m$ matrices Q and A as in the statement of the lemma. Let j be such that $\sigma(j) = m - 1$ (we have $1 < j < m$).

(a) Take any traverse of A. Assume that it contains the $(m-1, j)$-entry (this is the case of the traverse depicted in Fig. 3.5 with underlined entries). Suppressing the $m-1$st row and the jth column of A (and of $M_w + Q$), we are reduced to a similar case in S_{m-1}, which allows us to use induction.

(b) If the traverse does not contain the $(m-1, j)$-entry (this is the case of the traverse depicted in Fig. 3.5 with overlined entries), it necessarily contains the $(m-1, 1)$-entry which is an entry of Q, hence of A. We claim that it also contains all the entries in position $(\sigma(2), 2)$, $(\sigma(3), 3)$, ..., $(\sigma(j-1), j-1)$. Indeed, the only nonzero entries of A in the

$$\begin{pmatrix} * & * & \overline{*} & 1 \\ * & \overline{1} & 0 & 0 \\ \overline{*} & * & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 3.6

$\sigma(2)$th row are in the first and second columns. Since the traverse already contains the $(m-1, 1)$-entry, it has to contain the $(\sigma(2), 2)$-entry. Similarly, the only nonzero entries of A in the $\sigma(3)$th row are in the first, second, and third columns. Since the traverse already contains entries in position $(m-1, 1)$ and $(\sigma(2), 2)$, it has to contain the $(\sigma(3), 3)$-entry. This argument works up to the $\sigma(j-1)$st row; it shows that the traverse contains the entries indicated in the claim.

Suppress columns $1, 2..., j-1$ and rows $\sigma(1), \sigma(2), ..., \sigma(j-1)$ in the matrices $M_w + Q$ and A. What is left is a similar matrix of smaller size (Fig. 3.6 shows the matrix obtained by suppressing these rows and columns in the example of Fig. 3.5 with overlined entries), which allows us to apply the induction hypothesis. ∎

3.8. *The Matrices $\Pi_{\mathbf{i}}$.* In order to prove (3.2) when the ring R is not commutative, it will be useful to label the crossing points of the pseudo-line arrangement Γ of the reduced decomposition $\mathbf{i} = (i_1, ..., i_N)$ with double-indexed variables x_{rs} where $1 \leqslant r < s \leqslant n$. More precisely, we label the kth crossing point of Γ with the variable $x_{r_k s_k}$, where $r_k < s_k$ are the left labels of the two pseudo-lines intersecting at the kth crossing point. Since two pseudo-lines intersect in at most one point, the variables $x_{r_k s_k}$ labelling the crossing points are distinct. Observe also that the only variables x_{ij} coming up as labels for the crossing points of Γ are the ones with indices i, j such that $i < j$ and $w^{-1}(i) > w^{-1}(j)$.

We set

$$\Pi_{\mathbf{i}} = P_{i_1}(x_{r_1 s_1}) \cdots P_{i_N}(x_{r_N s_N}). \tag{3.5}$$

Alternatively, $\Pi_{\mathbf{i}}$ can be defined by induction on the length of $\mathbf{i}$ according to the following rules:

(a) if $\mathbf{i} = (i)$ is of length one, then $\Pi_{\mathbf{i}} = P_i(x_{i, i+1})$;

(b) if $\mathbf{i} = (i_1, ..., i_{N-1}, i_N)$ is of length $N > 1$, define $\mathbf{i}' = (i_1, ..., i_{N-1})$; if we set $w' = s_{i_1} \cdots s_{i_{N-1}}$, then

$$\Pi_{\mathbf{i}} = \Pi_{\mathbf{i}'} P_{i_N}(x_{w'(i_N), w'(i_N+1)}).$$

It follows from the definition of $\Pi_{\mathbf{i}}$ and of Proposition 3.2 that each entry of $\Pi_{\mathbf{i}}$ differing from 0 or from 1 (equivalently, in the diagram of w) is a sum of monomials of the form

$$x_{a_1a_2}x_{a_2a_3}\cdots x_{a_{q-1}a_q}, \tag{3.6}$$

where $a_1<a_2<\cdots<a_{q-1}<a_q$. Moreover, if it is the (i,j)-entry of $\Pi_{\mathbf{i}}$, we have

$$a_1=i \qquad \text{and} \qquad a_q=w(j). \tag{3.7}$$

3.9. *Proof of Theorem* 3.3: *The General Case.* It is enough to prove (3.2) when the matrix $M\in C_w$ is of the form $M=\Pi_{\mathbf{i}}$. Then (3.2) is equivalent to

$$x_{r_ks_k}=\varepsilon_k\,|M_{I_k,\,J_k}| \tag{3.8}$$

$(1\leqslant k\leqslant N)$. Start as in Subsection 3.4. Lemma 3.5 implies that the matrix $M_{I_k,\,J_k}$ is of the form $M_\sigma+Q$ considered in Lemma 3.7 for some $\sigma\in S_m$. We use this to prove the following claim.

Claim. Any monomial in any traverse of $M_{I_k,\,J_k}$ is of the form $x_{c_1c_2}x_{c_2c_3}\cdots x_{c_{t-1}c_t}$, where $c_1<c_2<\cdots<c_{t-1}<c_t$. Indeed, by Lemma 3.7, we know that any traverse of $M_{I_k,\,J_k}$ is of the form $M_{1,\,j_1}M_{\sigma(j_1),\,j_2}M_{\sigma(j_2),\,j_3}\cdots M_{\sigma(j_{p-1}),\,j_p}$ where the $M_{u,\,v}$ stand for the entries of $M_{I_k,\,J_k}$. The latter is a submatrix of $\Pi_{\mathbf{i}}$. Therefore, by (3.6) and (3.7), we have

$$M_{\sigma(j_u),\,j_{u+1}}=\sum x_{a_1a_2}x_{a_2a_3}\cdots x_{a_{q-1}a_q},$$

where the summation is over some $a_1<a_2<\cdots<a_{q-1}<a_q$ such that $a_1=\sigma(j_u)$ and $a_q=\sigma(j_{u+1})$. Similarly, for $M_{\sigma(j_{u+1}),\,j_{u+2}}$ we have

$$M_{\sigma(j_{u+1}),\,j_{u+2}}=\sum x_{b_1b_2}x_{b_2b_3}\cdots x_{b_{r-1}b_r}$$

with $b_1=\sigma(j_{u+1})$ and $b_r=\sigma(j_{u+2})$. Therefore, $b_1=a_q$, which shows that the product $M_{\sigma(j_u),\,j_{u+1}}M_{\sigma(j_{u+1}),\,j_{u+2}}$ is of the form (3.6). This argument extends to the whole product $M_{1,\,j_1}M_{\sigma(j_1),\,j_2}M_{\sigma(j_2),\,j_3}\cdots M_{\sigma(j_{p-1}),\,j_p}$, which proves the claim.

The above claim implies that $|M_{I_k,\,J_k}|$ is of the form

$$|M_{I_k,\,J_k}|=\sum\pm x_{c_1c_2}x_{c_2c_3}\cdots x_{c_{t-1}c_t},$$

where the indices are increasing: $c_1 < c_2 < \cdots < c_{t-1} < c_t$. By the proof in the commutative case (see Subsection 3.6), we know that $|M_{I_k, J_k}| \equiv \varepsilon_k x_{r_k s_k}$ modulo the ideal $\mathscr{I}$ generated by the commutators of the variables x_{ij}. Now, since the monomials (3.6) are linearly independent mod $\mathscr{I}$, the congruence

$$\sum_{c_1 < c_2 < \cdots < c_{t-1} < c_t} \pm x_{c_1 c_2} x_{c_2 c_3} \cdots x_{c_{t-1} c_t} \equiv \varepsilon_k x_{r_k s_k} \qquad \bmod \mathscr{I}$$

implies the equality

$$\sum_{c_1 < c_2 < \cdots < c_{t-1} < c_t} \pm x_{c_1 c_2} x_{c_2 c_3} \cdots x_{c_{t-1} c_t} = \varepsilon_k x_{r_k s_k}.$$

This proves (3.8), hence Theorem 3.3, in the general case. ∎

3.10. *Remark.* The sets of indices I_k and J_k entering the statement of Theorem 3.3 can be recovered from the matrix $P_{\mathbf{i}}(x_1, ..., x_N)$ as follows. Take the entry of $P_{\mathbf{i}}(x_1, ..., x_N)$ whose linear term is x_k. We denote it by Φ_k. Let V_k be the set of those variables $x_1, ..., x_N$ that occur in Φ_k. Then I_k (resp. J_k) is the set of indices of the rows (resp. of the columns) of $Q_{\mathbf{i}}(x_1, ..., x_N)$ where the variables in V_k appear.

4. BALANCED LABELLINGS

The purpose of this section is to prove Theorem 1.2. To this end, we need the definition of a balanced labelling of the diagram D_w of a permutation $w \in S_n$.

Given an element (i, j) of D_w, we call *arm* (resp. *leg*) of (i, j) the subset of D_w consisting of all elements (i, k) with $k > j$ (resp. of all elements (h, j) with $h > i$). The *hook* of (i, j) is the union of $\{(i, j)\}$, of its arm and of its leg. An injective labelling L of D_w by the integers $1, 2, ..., N$ is said to be *balanced* if for any element of D_w, whose label we denote by a, the number of labels $> a$ in the leg is equal to the number of labels $< a$ in the arm.

This definition of a balanced labelling of a diagram extends the concept of balanced tableau introduced by Edelman and Greene [EG] for the Ferrers diagram of an integer partition. An injective balanced labelling in our sense is equivalent, up to transposition, to the definition given by Fomin *et al.* in [FGRS, Sect. 2].

In the Introduction, we associated a labelling $L_{\mathbf{i}}$ of the Rothe diagram D_w to any reduced decomposition $\mathbf{i}$ of w.

4.1. PROPOSITION. *The labelling $L_{\mathbf{i}}$ is injective and balanced.*

Proof. (a) From Proposition 3.2 it is clear that each variable $x_1, ..., x_N$, where N is the length of w, appears in the linear part $Q_{\mathbf{i}}(x_1, ..., x_N)$ of the matrix $P_{\mathbf{i}}(x_1, ..., x_N)$ and that distinct variables appear in distinct entries of $Q_{\mathbf{i}}(x_1, ..., x_N)$. Since D_w has N elements, it follows that the labelling $L_{\mathbf{i}}$ is injective.

(b) We shall show that $L_{\mathbf{i}}$ is balanced by constructing for each element (i, j) of D_w a one-to-one correspondence between the labels in the leg of (i, j) that are greater than the label of (i, j) and the labels in the arm of (i, j) that are smaller than the label of (i, j).

Let a be the label of (i, j) in $L_{\mathbf{i}}$. Let (i', j) be an element of the leg of (i, j) with label b with $b > a$. By definition of the leg, we have $i' > i$. Let us consider the pseudo-line arrangement Γ of $\mathbf{i}$ with its crossings points labelled from left to right by x_1 up to x_N. By Proposition 3.2 and the definition of $L_{\mathbf{i}}$, the variable x_a labels the intersection of the pseudo-line with left label i and of the pseudo-line with right label j. Similarly, the variable x_b labels the intersection of the pseudo-line with left label i' and of the pseudo-line with right label j. Since $b > a$, the crossing point labelled x_b lies to the right of the crossing point labelled x_a on the pseudo-line with right label j. Since the left labels i and i' verify $i' > i$ and since pseudo-lines intersect in at most one point, the pseudo-line with left label i' must intersect the pseudo-line with left label i in a point, labelled, say, by x_c, lying left of the crossing point labelled x_a on the pseudo-line with left label i (see Fig. 4.1, which shows the only possible configuration). It follows that $c < a$ and $w^{-1}(i') = j' > j$. Therefore, to any label $b > a$ of an element (i', j) in the leg of (i, j) we assigned an element $(i, j') = (i, w^{-1}(i'))$ in the arm of (i, j) with label $c < a$.

Proceeding in a similar way, to any label $c < a$ of an element (i, j') in the arm of (i, j) one assigns an element $(w(j'), j)$ in the leg of (i, j) with label $b > a$. These two maps are inverse of each other. ∎

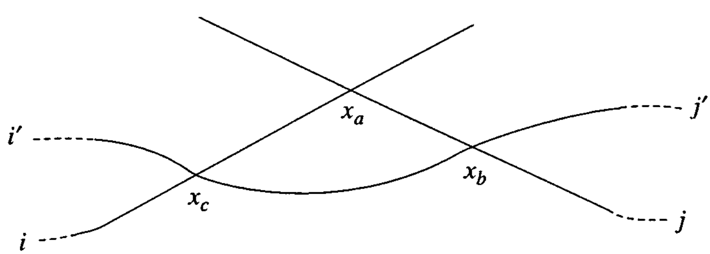


FIGURE 4.1

So far we have a map $\mathbf{i} \mapsto L_{\mathbf{i}}$ from the set $\mathscr{R}(w)$ of reduced decompositions of w to the set of injective balanced labellings of D_w. The fact that this map is bijective may be deduced from [FGRS, Theorem 2.4]. For the sake of completeness, we shall give a self-contained proof of this fact after a few preliminaries.

Fix a permutation $w \in S_n$ of length $N \geqslant 1$ and an injective labelling L of the Rothe diagram D_w of w. Let $(i, j) \in D_w$ be the element with label N. We assume that L is balanced. This assumption implies the following lemma.

4.2. LEMMA. (a) *We have* $w(j+1) = i$.

(b) *If* $(k, j) \in D_w$ *for some* $k < i$, *then* $(k, j+1) \in D_w$.

(c) *If* a_k *is the label of* $(k, j) \in D_w$ *with* $k < i$ *and* b_k *is the label of* $(k, j+1)$, *then* $a_k > b_k$.

Proof. (a) Since N is the greatest label, there is no greater label in the leg of (i, j). The labelling L being balanced, there is no smaller label in the arm of (i, j). This means that D_w does not contain any element (i, j') with $j' > j$. If we had $w(j+1) > i$, then, since $(i, j) \in D_w$, we would have $w^{-1}(i) > j$; as $w^{-1}(i) \neq j+1$, we would have $w^{-1}(i) > j+1$, which would imply $(i, j+1) \in D_w$ and contradict our hypothesis on (i, j).

We now show that $w(j+1) < i$ is impossible as well. Indeed, by (1.1), the assumption $(i, j) \in D_w$ implies that $(w(j+1), j) \in D_w$. The arm of $(w(j+1), j)$ is empty by (1.1). By the assumption on L, there is no label in the leg of $(w(j+1), j)$ that is greater than the label of $(w(j+1), j)$. But $w(j+1) < i$ implies that (i, j), which has the greatest label, is in the leg of $(w(j+1), j)$. This is a contradiction.

(b) In order to show that $(k, j+1) \in D_w$, we have to check that $k < w(j+1)$ and $j+1 < w^{-1}(k)$. The first inequality follows from $k < i$ and Part (a). Let us prove the second inequality: $(k, j) \in D_w$ implies that $j < w^{-1}(k)$. So it is enough to check that $w^{-1}(k) \neq j+1$. This follows from $k \neq i$ and $j+1 = w^{-1}(i)$ (Part (a)).

(c) Suppose Statement (c) is not true. Then there is a maximal $k < i$ such that $a_k < b_k$. Assume that there are exactly p labels $> a_k$ in the leg of (k, j). Since $N > a_k$, we have $p \geqslant 1$. By the assumption on the labelling L, there are p labels $u_1, ..., u_p$, all $< a_k$, in the arm of (k, j). Since $u_i < a_k < b_k$ for all $i = 1, ..., p$, then $u_1, ..., u_p$ are labels $< b_k$ in the arm of $(k, j+1)$, hence there are p labels $> b_k$ in the leg of $(k, j+1)$. We may assume by Part (a) they label elements $(k', j+1)$ of D_w such that $k < k' < i$. Assume that these labels are $b_{h_1}, ..., b_{h_p}$. Then for any $i = 1, ..., p$, we have $a_{h_i} > b_{h_i} > b_k > a_k$ by the maximality of k. The label N together with $a_{h_1}, ..., a_{h_p}$ form $p+1$ labels $> a_k$ in the leg of (k, j), which is in contradiction with the definition of p. ∎

Let $w' = ws_j$, where again (i, j) is the element of D_w with label N.

4.3. LEMMA. *The permutation w' is of length $N-1$. Moreover, the Rothe diagram $D_{w'}$ is obtained from D_w by first removing the element (i, j), then by exchanging the jth and the $j+1$st columns.*

Proof. Lemma 4.2(a) shows that we are in the situation of Fig. 2.1a. Lemma 4.3 follows. ∎

In view of Lemmas 4.2 and 4.3, out of the labelling L we get an injective labelling L' of $D_{w'}$ by removing the label N and by exchanging the jth and the $j+1$st columns. As a consequence of Lemma 4.2(c), we get the following result, which is crucial for the proof of Theorem 1.2.

4.4. Corollary. *The labelling L' of $D_{w'}$ is balanced.*

4.5. *Proof of Theorem* 1.2. It is clear when w is of length 1, i.e., when w is a simple transposition s_j for some $1 \leqslant j \leqslant n-1$.

To a permutation $w \in S_n$ of length $N \geqslant 1$ and an injective balanced labelling L of D_w, we assign a reduced decomposition $\mathbf{i}$ of w as follows.

Let $(i, j) \in D_w$ be the element with label N. As above, we introduce the permutation $w' = ws_j$, which is of length $N-1$ (Lemma 4.3), and the labelling L' of $D_{w'}$ obtained from L by removing the label N and by exchanging the jth and the $j+1$st columns. The labelling L' is balanced by Corollary 4.4.

Starting the same procedure with the couple (w', L'), we get a new couple (w'', L''), where $w'' = w's_{j'}$ is a permutation of length $N-2$, the integer j' is the number of the column in which the highest label $N-1$ of L' appears, and L'' is the balanced labelling of $D_{w''}$ obtained from L' by removing the label $N-1$ and by exchanging the j'th and the $j+1$st columns.

Iterating this procedure, we get a sequence of permutations w_N, w_{N-1}, w_{N-2}, ..., w_0, where w_i is of length i for all $i = 0, ..., N$, and a sequence of integers $j_N, j_{N-1}, j_{N-2}, ..., j_1$ with $w_N = w, w_{N-1} = w', w_{N-2} = w''$, and $j_N = j, j_{N-1} = j', j_{N-2} = j''$, such that $w_i = w_{i-1} s_{j_i}$, hence $w_i = s_{j_1} s_{j_2} \cdots s_{j_i}$ for all $i = 1, ..., N$. In particular,

$$w = w_N = s_{j_1} s_{j_2} \cdots s_{j_N},$$

which implies that $(j_1, j_2, ..., j_N)$ is a reduced decomposition of w. We denote this reduced decomposition by $r(L)$.

This defines a map r from the set of injective balanced labellings of D_w to the set $\mathscr{R}(w)$ of reduced decompositions of w. It is easy to check that r is a two-sided inverse to the map $\mathbf{i} \mapsto L_{\mathbf{i}}$ of Theorem 1.2. ∎

$$\begin{pmatrix} 1 & 3 & 7 & 5 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & 8 & 6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \underline{9} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 3 & 5 & 7 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & 6 & \underline{8} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{4} \begin{pmatrix} 1 & 3 & 5 & 2 & \underline{7} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & 6 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{5}$$

$$\begin{pmatrix} 1 & 3 & 5 & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & \underline{6} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 3 & 2 & \underline{5} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{4} \begin{pmatrix} 1 & 3 & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \underline{4} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{2}$$

$$\begin{pmatrix} 1 & 2 & \underline{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & \underline{2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{2} \begin{pmatrix} \underline{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \xrightarrow{1} \emptyset.$$

FIGURE 4.2

Figure 4.2 shows how $r(L)$ is obtained using the above procedure when we start from the following injective balanced labelling of $w = 2\,4\,6\,5\,3\,1$ (the crosses indicate the entries of the permutation matrix M_w):

$$L = \begin{pmatrix} 1 & 3 & 7 & 5 & 2 & \times \\ \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & 8 & 6 & \times & \cdot \\ \cdot & \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 9 & \times & \cdot & \cdot \\ \cdot & \cdot & \times & \cdot & \cdot & \cdot \end{pmatrix} \tag{4.1}$$

In each labelling of Fig. 4.2 we underlined the greatest label. The integer j that appears when we pass from one labelling to one with one label less is written over the corresponding arrow. The last labelling is empty because it is the labelling of the identity permutation. For this example, we get $r(L) = (1, 2, 3, 2, 4, 3, 5, 4, 3)$.

4.6. *Remark.* The bijection of Theorem 1.2 is the same as the one described for the longest permutation $w_0 \in S_n$ in [EG] and, up to transposition, for general w in [FGRS].

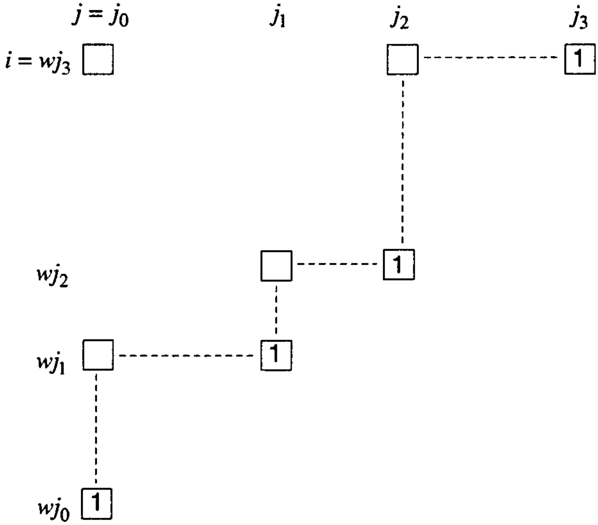


FIGURE 4.3

4.7. *Remark.* The matrix $P_{\mathbf{i}}(x_1, ..., x_N)$ can be recovered from the labelling $L_{\mathbf{i}}$ as follows. If we denote the (k, l)-entry of $L_{\mathbf{i}}$ by $v(k, l)$, then the (i, j)-entry of $P_{\mathbf{i}}(x_1, ..., x_N)$ for $(i, j) \in D_w$, is given by

$$\sum x_{v(w(j_k),\, j_{k-1})} \cdots x_{v(w(j_2),\, j_1)} x_{v(w(j_1),\, j_0)}, \tag{4.2}$$

where the sum runs over all indices such that $k \geqslant 1$, $j = j_0 < j_1 < \cdots < j_k = w^{-1}(i)$, $w_j = w(j_0) > w(j_1) > \cdots > w(j_k) = i$, and $v(w(j_k), j_{k-1}) < \cdots < v(w(j_2), j_1) < v(w(j_1), j_0)$. See Fig. 4.3 for a graphical interpretation of the conditions on the indices j_p.

4.8. *An Extension of Theorem* 1.2. Transposing from [FGRS], we say that a labelling of D_w, with *not necessarily distinct* labels, is *balanced* if each hook is balanced in the following sense: we say that a hook is balanced if the corner label remains unchanged after we have rearranged the labels in the hook so that they weakly increase upwards and from left to right. When the labels are distinct, this definition coincides with the one we gave at the beginning of this section. We also say that a labelling is *row-strict* if the labels in each row are distinct. Then the following extension of Theorem 1.2 holds: the set of row-strict balanced labellings of D_w coincides with the set of linear parts of the matrix products $P_{\mathbf{i}}(x_1, x_2, ..., x_N)$ for all reduced decompositions $\mathbf{i} = (i_1, i_2, ..., i_N)$ of w and all totally ordered variables $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$, the equality $x_k = x_{k+1}$ being permitted only if $i_k > i_{k+1}$.

Let us end this section by showing how to recover the labelling $R_{\mathbf{i}}$, defined by (1.11), from the labelling $L_{\mathbf{i}}$.

4.9. LEMMA. *The (i, j)-entry of $R_{\mathbf{i}}$ is the number of labels in the arm of (i, j) in $L_{\mathbf{i}}$, smaller than the label $v(i, j)$ of (i, j); equivalently, it is the number of labels in the leg of (i, j) in $L_{\mathbf{i}}$, greater than the label $v(i, j)$.*

Proof. The labelling $R_{\mathbf{i}}$ counts the quadratic monomials in $P_{\mathbf{i}}(x_1, ..., x_N)$, hence, by Proposition 3.2, the efficient pseudo-lines in the pseudo-line arrangement of $\mathbf{i}$. One then argues as in Part (b) of the proof of Proposition 4.1 so as to conclude that each efficient pseudo-line produces a label in the arm that is smaller than the label in the (i, j)-position, and a label in the leg that is greater than it; conversely, such labels come from efficient pseudo-lines. ∎

Applying Lemma 4.9 to the labelling $L_{\mathbf{i}}$ given by (4.1), we get

$$R_{\mathbf{i}} = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

4.10. *Remark.* Recall that the code (also called the Lehmer code) of a permutation $\sigma \in S_p$ is the vector $(c_1, ..., c_p) \in \mathbf{N}^p$ such that $c_i = \text{card}\{ j \mid j > i$ and $\sigma(j) < \sigma(i)\}$. The integer c_i is also the number of elements in the ith column of the diagram D_σ. The code function is a bijection between S_p and the set of vectors $(c_1, ..., c_p) \in \mathbf{N}^p$ such that $c_1 \leqslant p-1$, $c_2 \leqslant p-2$, ..., $c_p \leqslant 0$.

Lemma 4.9 states that each row of $R_{\mathbf{i}}$ is the code of the permutation induced by the natural order on the labels in the corresponding row in $L_{\mathbf{i}}$. Similarly, one interprets the columns as codes of permutations, the inversions being now counted with respect to the reverse order on labels: e.g., $c = (1, 2, 0, 0)$ codes the permutation 3 1 4 2.

5. CHARACTERIZING COMMUTATION CLASSES OF REDUCED DECOMPOSITIONS

The purpose of this section is to prove Theorem 1.3. Recall the matrices $\Pi_{\mathbf{i}}$ of Subsection 3.

5.1. THEOREM. *Two reduced decompositions* $\mathbf{i}$ *and* $\mathbf{j}$ *of a permutation* w *belong to the same commutation class if and only if* $\Pi_{\mathbf{i}} = \Pi_{\mathbf{j}}$.

Proof. (a) If $\mathbf{i}$ and $\mathbf{j}$ differ by a 2-move, then Relation (1.4a) and the definition of $\Pi_{\mathbf{i}}$ and $\Pi_{\mathbf{j}}$ imply that $\Pi_{\mathbf{i}} = \Pi_{\mathbf{j}}$. This can also be seen using the pseudo-line arrangements of $\mathbf{i}$ and $\mathbf{j}$ (which are isotopic as configurations in the plane) and Proposition 3.2. It follows that if $\mathbf{i}$ and $\mathbf{j}$ belong to the same commutation class, then $\Pi_{\mathbf{i}} = \Pi_{\mathbf{j}}$.

(b) Let $\mathbf{i}$ and $\mathbf{j}$ be reduced decompositions of w such that $\Pi_{\mathbf{i}} = \Pi_{\mathbf{j}}$. We wish to show that $\mathbf{i}$ and $\mathbf{j}$ belong to the same commutation class. We shall proceed by induction on the length of w.

If w is of length 1, then it has exactly one reduced decomposition, hence exactly one commutation class and there is nothing to show.

Suppose that $\mathbf{i}$ and $\mathbf{j}$ are of length $N > 1$. Consider their pseudo-line arrangements denoted by $\Gamma_{\mathbf{i}}$ and $\Gamma_{\mathbf{j}}$, respectively. Let us concentrate on the leftmost crossing point of $\Gamma_{\mathbf{i}}$ and the two pseudo-lines intersecting at this point. Their left labels are necessarily of the form i_1 and $i_1 + 1$, where i_1 is the first index in $\mathbf{i}$. By definition of $\Pi_{\mathbf{i}}$, the double-indexed variable x_{i_1, i_1+1} appears in the matrix $\Pi_{\mathbf{i}}$, hence in the matrix $\Pi_{\mathbf{j}}$. This means that in $\Gamma_{\mathbf{j}}$ the pseudo-line L with left label i_1 and the pseudo-line L' with left label $i_1 + 1$ intersect each other. Let C be the union of the connected components of the plane deprived of $\Gamma_{\mathbf{j}}$ situated above L and under L', to the left of their joint intersection. See Fig. 5.1.

We claim that there is no pseudo-line of $\Gamma_{\mathbf{j}}$ crossings C. It follows from the claim that $\Gamma_{\mathbf{j}}$ can be isotoped to a pseudo-lined arrangement whose leftmost crossing point is labelled by x_{i_1, i_1+1}. Equivalently, one can pass through a sequence of 2-moves from $\mathbf{j}$ to a reduced decomposition $\mathbf{k}$ with first index $k_1 = i_1$. By Part (a), we have $\Pi_{\mathbf{k}} = \Pi_{\mathbf{j}} = \Pi_{\mathbf{i}}$.

Let $\mathbf{i}'$ and $\mathbf{k}'$ be the reduced decompositions obtained respectively from $\mathbf{i}$ and $\mathbf{k}$ by removing the first index $i_1 = k_1$. The equality $\Pi_{\mathbf{i}} = \Pi_{\mathbf{k}}$ implies $\Pi_{\mathbf{i}'} = \Pi_{\mathbf{k}'}$. Since $\mathbf{i}'$ and $\mathbf{k}'$ are of length $N-1$, we may appeal to the induction hypothesis and conclude that $\mathbf{i}'$ and $\mathbf{k}'$ belong to the same commutation class. So do $\mathbf{i}$ and $\mathbf{k}$, hence $\mathbf{i}$ and $\mathbf{j}$.

We now prove the claim. Suppose that some pseudo-line L'' of $\Gamma_{\mathbf{j}}$ with left label r (necessarily $\neq i_1, i_1+1$) crosses the region C. Then it must

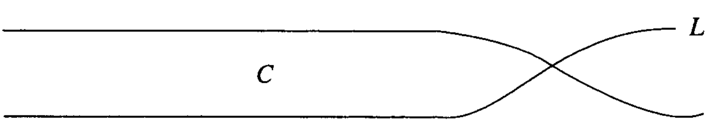


FIGURE 5.1

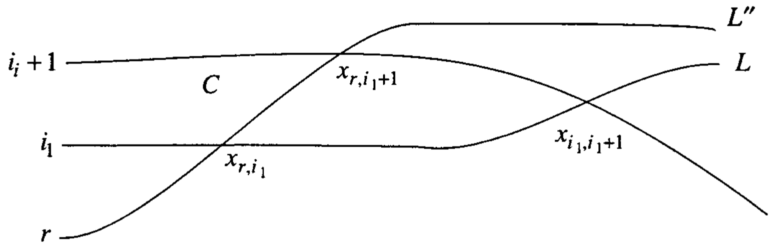


FIG. 5.2. The area C in $\Gamma_{\mathbf{j}}$ when $r<i_1$.

necessarily intersect L and L' to the left of the intersection of L and L'. There are two cases: either $r<i_1$ or $r>i_1+1$.

Suppose that $r<i_1$. The intersection of L'' with L produces a crossing point with label x_{r,i_1} whereas the intersection of L'' with L' produces a crossing point with label x_{r,x_1+1} (see Fig. 5.2). By Proposition 3.2 the matrix $\Pi_{\mathbf{j}}$ contains simultaneously the monomials x_{r,i_1+1} and $x_{r,i_1}x_{i_1,i_1+1}$. Therefore, the matrix $\Pi_{\mathbf{i}}$ must contain the same monomials. Since the pseudo-lines L, L', L'' intersect pairwise in $\Gamma_{\mathbf{j}}$, they must intersect pairwise in $\Gamma_{\mathbf{i}}$. Because the crossing point labelled x_{i_1,i_1+1} is the leftmost in $\Gamma_{\mathbf{i}}$, the crossing point labelled x_{r,i_1+1} comes before the crossing point labelled x_{r,i_1} (see Fig. 5.3). By Proposition 3.2 this implies that the quadratic monomial $x_{r,i_1}x_{i_1,i_1+1}$ cannot appear in the matrix $\Pi_{\mathbf{i}}$, which leads to a contradiction.

The case $r>i_1+1$ is treated in a similar way (see Figs. 5.4 and 5.5). ∎

The previous proof also shows that the following holds. The equivalence of (i) and (ii) is well-known.

5.2. COROLLARY. *For two reduced decompositions* $\mathbf{i}$ *and* $\mathbf{j}$ *of* w, *the following conditions are equivalent*:

(i) $\mathbf{i}$ *and* $\mathbf{j}$ *belong to the same commutation class*;

(ii) *their pseudo-line arrangements are isotopic as configurations in the plane*;

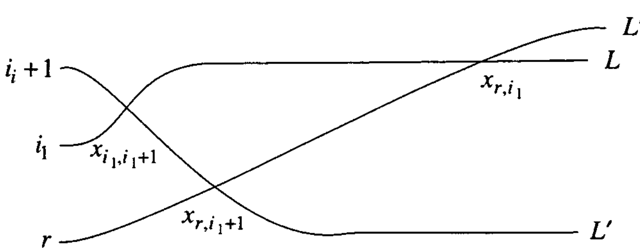


FIG. 5.3. The leftmost part of $\Gamma_{\mathbf{i}}$ when $r<i_1$.

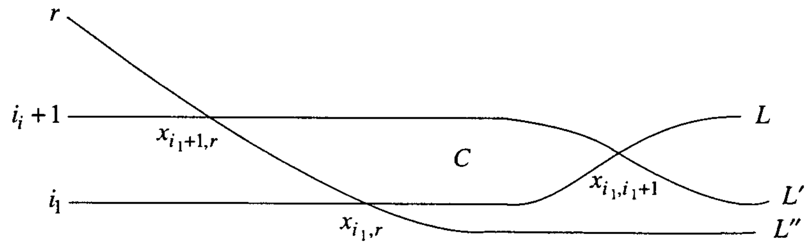


FIG. 5.4. The area C in $\Gamma_{\mathbf{j}}$ when $r > i_1 + 1$.

(iii) *their pseudo-line arrangements have the same efficient pseudo-lines*;

(iv) *the matrices $\Pi_{\mathbf{i}}$ and $\Pi_{\mathbf{j}}$ have the same quadratic part.*

As a consequence, the matrix $\Pi_{\mathbf{i}}$ is completely determined by its quadradic part.

5.3. *Proof of Theorem* 1.3. If $\mathbf{i}$ and $\mathbf{j}$ belong to the same commutation class, then $\Pi_{\mathbf{i}} = \Pi_{\mathbf{j}}$ by Theorem 5.1. It follows that these matrices have the same number of quadratic terms in each entry; hence, so have the matrices $P_{\mathbf{i}}(X, ..., X)$ and $P_{\mathbf{j}}(X, ..., X)$. In view of (1.11), we get $R_{\mathbf{i}} = R_{\mathbf{j}}$.

Conversely, if we know $R_{\mathbf{i}}$, then, since quadratic terms of $P_{\mathbf{i}}$ correspond to efficient pseudo-lines, and since efficient pseudo-lines give smaller labels in the arm in the matrix $L_{\mathbf{i}}$, we know how many labels in the arm of any hook are smaller than the corner label. Since a permutation is determined by its code, this in turn implies that we know the relative order of the labels in each row of the diagram. Hence we can determine the labels that are smaller than a given corner label, hence the efficient pseudo-lines. Therefore, if $R_{\mathbf{i}} = R_{\mathbf{j}}$, then $\mathbf{i}$ and $\mathbf{j}$ have the same efficient pseudo-lines, which by Corollary 5.2 implies that $\mathbf{i}$ and $\mathbf{j}$ belong to the same commutation class. ∎

5.4. Corollary. *A reduced decomposition* $\mathbf{i}$ *of* w *is in the same commutation class as the canonical sequence of* w *if and only if the labelling* $R_{\mathbf{i}}$ *consists of zeros only.*

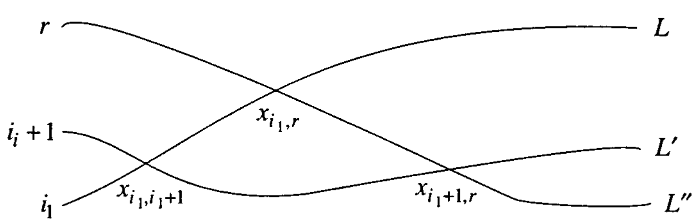


FIG. 5.5. The leftmost part of $\Gamma_{\mathbf{i}}$ when $r > i_1 + 1$.

Proof. By Proposition 2.1, if $\mathbf{j}$ is the canonical sequence of the permutation w, then the matrix $P_{\mathbf{j}}(x_1, ..., x_n)$ has no quadratic monomials. Therefore, $R_{\mathbf{j}} = 0$. We conclude with Theorem 1.3. ∎

5.5. *Characterization of the Matrices* $R_{\mathbf{i}}$. If $k_1, ..., k_l$ are the labels in a row of $R_{\mathbf{i}}$, then $(k_1, ..., k_l)$ is by Lemma 4.9 the code of a permutation in S_l, which we call the row permutation of this row. Similarly, we have a column permutation (the code being interpreted with respect to the decreasing order on integers). Thus, to each matrix $R_{\mathbf{i}}$, we associate a *bipermutation matrix* with support in the diagram of w, whose (i, j)-entry consists of the couple (α, β), where α (resp. β) is the corresponding digit of the ith row (resp. jth) column) permutation of $R_{\mathbf{i}}$. From this, we define a graph whose vertices are the entries of the bipermutation matrix, with edges $u \to v$ if either u, v are in the same row and $u = (\alpha, \beta)$ and $v = (\alpha + 1, \beta')$, or u, v are in the same column and $u = (\alpha, \beta)$ and $v = (\alpha', \beta + 1)$. An example of a labelling $R_{\mathbf{i}}$, corresponding to (1.10), is given in Fig. 5.6 together with its bipermutation matrix and its graph.

5.6. Theorem. *The mapping $\mathbf{i} \mapsto R_{\mathbf{i}}$ defines a bijection from the set of commutation classes of reduced decompositions of w onto the set of labellings of D_w by natural numbers such that*

(i) *the kth element in each row* (*resp. each column*) *is $\leqslant l - k$, where l is the length of the row* (*resp. the column*), *and*

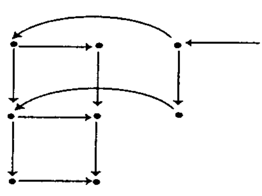
(ii) *the previously described graph has no cycles.*

Proof. Let us show that $R_{\mathbf{i}}$ satisfies the two conditions: for the first one, it is an immediate consequence of the reminder at the beginning of Subsection 5.5; for the second one, note that the labels in $L_{\mathbf{i}}$ define a total order which is compatible with the graph. Therefore, the latter has no cycle.

Conversely, let R be a labelling of D_w satisfying (i) and (ii). The graph having no cycle, we can find a compatible labelling of D_w by the numbers $1, 2, ..., N = l(w)$. This labelling is balanced by construction. Therefore, it comes from a reduced decomposition of w, and so does R. ∎

$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & \cdot \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} (3,1) & (4,1) & (2,1) & (1,1) \\ (2,2) & (3,2) & (1,2) & \cdot \\ (1,3) & (2,3) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$R_{\mathbf{i}}$ bipermutation



graph

FIGURE 5.6

5.7. *Remark.* In analogy with (4.2), we can express the matrix $\Pi_{\mathbf{i}}$ in terms of the bipermutation matrix described previously or, equivalently, of the associated graph. Indeed, denote by $\succ$ the partial order on the entries of D_w induced by this graph. Then, for (i, j) in D_w, the (i, j)-entry of $\Pi_{\mathbf{i}}$ is given by

$$\sum x_{w(j_k),\, w(j_{k-1})} \cdots x_{w(j_2),\, w(j_1)} x_{w(j_1),\, w(j_0)},$$

where the sum runs over all indices such that $k \geqslant 1$,

$$j = j_0 < j_1 < \cdots < j_k = w^{-1}(i), \qquad w(j) = w(j_0) > w(j_1) > \cdots > w(j_k) = i,$$

and $(w(j_1), j_0) \succ (w(j_2), j_1) \succ \cdots \succ (w(j_k), j_{k-1})$.

6. THE POSET STRUCTURE OF THE SET OF COMMUTATION CLASSES

Let $w \in S_n$ be a permutation of length N and $\mathscr{R}(w)$ be the set of reduced decompositions of w. Recall that $\mathbf{i}, \mathbf{j} \in \mathscr{R}(w)$ belong to the same commutation class if they can be obtained from each other by a sequence of 2-moves. We denote by $\mathscr{C}(w)$ the set of commutation classes in $\mathscr{R}(w)$. Manin and Schechtman [MS] put a partial order on $\mathscr{C}(w)$: it is the reflexive and transitive binary relation $\leqslant_c$ generated by

$$(..., i+1, i, i+1, ...) <_c (..., i, i+1, i, ...)$$

for all $1 \leqslant i \leqslant N-1$. (That this order is well defined may be proved by considering the sum of the indices in the reduced decompositions.)

By [MS, BZ, LZ], the poset $\mathscr{C}(w)$ has a unique minimal element. We show that it is the commutation class α_w of the canonical sequence $(j_1, ..., j_N)$ of w, as defined in Section 2.

6.1. Proposition. *We have $\alpha_w \leqslant_c \beta$ for any element β of $\mathscr{C}(w)$.*

Proof. Let $\mathbf{i} = (..., i, i+1, i, ...)$ and $\mathbf{j} = (..., i+1, i, i+1, ...)$ be reduced decompositions of w differing by a 3-move. This means that the respective pseudo-line arrangements $\Gamma_{\mathbf{i}}$ and $\Gamma_{\mathbf{j}}$ differ only locally as shown in Fig. 6.1. Let $k < l < m$ be the left labels of the three pseudo-lines appearing in the parts of $\Gamma_{\mathbf{i}}$ and $\Gamma_{\mathbf{j}}$ shown in Fig. 6.1. From Proposition 3.2 it is clear that all entries of the matrices $\Pi_{\mathbf{i}}$ and $\Pi_{\mathbf{j}}$ of Subsection 3.8 are the same, except the $(k, w^{-1}(m))$-entries. The pseudo-line with left label l is efficient for the roof of $\Gamma_{\mathbf{i}}$ whose top is labelled by x_{km}; but the same pseudo-line is not efficient for the corresponding roof in $\Gamma_{\mathbf{j}}$. Therefore, $\Pi_{\mathbf{j}}$ has one quadratic monomial

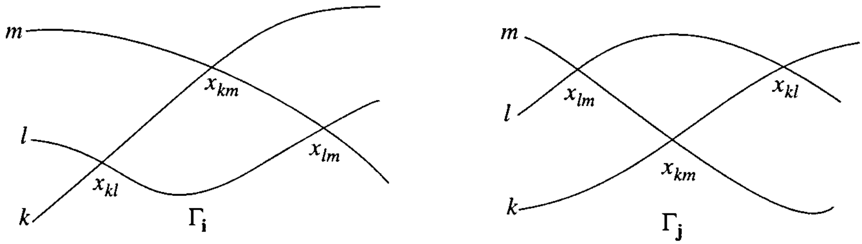


FIGURE 6.1

less than $\Pi_{\mathbf{i}}$, namely $x_{kl}x_{lm}$, and the only change when passing from the labelling $R_{\mathbf{i}}$ to the labelling $R_{\mathbf{j}}$ is that one label of $R_{\mathbf{i}}$ diminishes by 1.

Fix an element $\beta \in \mathscr{C}(w)$ represented by a reduced decomposition $\mathbf{i}$ of w. Suppose it is not in the commutation class of α_w. Then $R_{\mathbf{i}} \neq 0$ by Corollary 5.4. This means that the pseudo-line arrangement $\Gamma_{\mathbf{i}}$ of $\mathbf{i}$ has a pseudo-line L that is efficient for some roof of $\Gamma_{\mathbf{i}}$, as in the left part of Fig. 6.1. We take L such that the triangle formed by L and the slopes of the roof is not crossed by any other pseudo-line (we may call this a region of minimal area; such regions always exist when there is an efficient pseudo-line). Let us move L in order to get the configuration shown in the right part of Fig. 6.1. For $\mathbf{i}$, this amounts to making 3-move $(..., i, i+1, i, ...) \rightarrow (..., i+1, i, i+1, ...)$ as well as some 2-moves. It follows from the definition of the partial order in $\mathscr{C}(w)$ that the commutation class of the new reduced decomposition $\mathbf{j}$ we have obtained by performing these moves is smaller than the commutation class of $\mathbf{i}$ for $<_c$. By the above discussion, the total sum of the labels in $R_{\mathbf{j}}$ is less than the total sum of the labels in $R_{\mathbf{i}}$. We iterate this procedure until we get a labelling $R=0$ for which we apply Corollary 5.4. In this way, we have found a sequence of classes $\beta_0, \beta_1, ..., \beta_p$ such that $\beta = \beta_0 >_c \beta_1 >_c \cdots >_c \beta_p = \alpha_w$. ∎

6.2. *Remark.* As observed in the previous proof, a 3-move as in Fig. 6.1 (from left to right) suppresses the monomial $x_{kl}x_{lm}$ and all its multiples from the matrix $\Pi_{\mathbf{i}}$. A 2-move does not change the matrix $\Pi_{\mathbf{i}}$. Note that one can also easily describe the effect of 2- and 3-moves on the matrix $P_{\mathbf{i}}(x_1, ..., x_N)$, thereby describing the polynomial change of parametrizations of the Schubert cell C_w. A 2-move in position $k, k+1$ of $\mathbf{i}$ interchanges the variables x_k and x_{k+1} in $P_{\mathbf{i}}(x_1, ..., x_N)$. A 3-move $s, s+1, s \mapsto s+1, s, s+1$ in position $k-1, k, k+1$ interchanges the variables x_{k-1} and x_{k+1} in $P_{\mathbf{i}}(x_1, ..., , x_N)$ (hence also in the balanced labelling $L_{\mathbf{i}}$), and suppresses the monomial $x_{k-1}x_{k+1}$ and its multiples.

The poset $\mathscr{C}(w)$ has a unique maximal element (see [BZ, LZ, MS]). Let us describe it. Let w_0 be the longest element of S_n, defined by $w_0(i) = n+1-i$ for all $i = 1, ..., n$. If $\mathbf{i} = (i_1, ..., i_N)$ is a reduced decomposition of w, then $\mathbf{i}^* = (n - i_1, ..., n - i_N)$ is a reduced decomposition of $w_0 w w_0$. It is clear that the map $\mathbf{i} \mapsto \mathbf{i}^*$ from $\mathscr{R}(w)$ to $\mathscr{R}(w_0 w w_0)$ is a bijection and preserves the commutation classes. It therefore induces a bijection $\beta \mapsto \beta^*$ from $\mathscr{C}(w)$ to $\mathscr{C}(w_0 w w_0)$. Let ω_w be the element of $\mathscr{C}(w)$ such that $\omega_w^* = \alpha_{w_0 w w_0}$ is the minimal element of $\mathscr{C}(w_0 w w_0)$.

6.3. Proposition. *We have $\beta \leqslant_c \omega_w$ for any element β of $\mathscr{C}(w)$.*

Proof. The definitions of $\leqslant_c$ and of the involution $*$ imply $\beta \leqslant_c \gamma \Rightarrow \gamma^* \leqslant_c \beta^*$. Hence $\beta \leqslant_c \omega_w$ is equivalent to $\omega_w^* = \alpha_{w_0 w w_0} \leqslant_c \beta^*$, which follows from Proposition 6.1. ∎

As an application, we characterize the *fully commutative* permutations, i.e., the permutations w such that the set $\mathscr{C}(w)$ consists of a single element (see, e.g., [BJS, Fa, St]). By Propositions 6.1 and 6.3, w is fully commutative if and only if $\alpha_w = \omega_w$.

By Theorem 1.3 the labelling $R_{\mathbf{i}}$ defined for any reduced decomposition $\mathbf{i}$ depends only on the commutation class β of $\mathbf{i}$. We denote it by R_β.

6.4. Theorem. *For any $w \in S_n$, the following statements are equivalent.*

(i) *The permutation w is fully commutative.*

(ii) *The labelling R_β for any commutation class $\beta \in \mathscr{C}(w)$ consists of zeroes only.*

(iii) *The labelling R_{ω_w} for the maximal element ω_w consists of zeroes only.*

(iv) *There is no efficient pseudo-line in any pseudo-line arrangement of w.*

Proof. (i) $\Rightarrow$ (ii). Since card $\mathscr{C}(w) = 1$, we have $\beta = \alpha_w$, hence $R_\beta = R_{\alpha_w} = 0$ by Corollary 5.4.

(ii) $\Rightarrow$ (iii). This is clear.

(iii) $\Rightarrow$ (i). In view of Theorem 1.3, $R_{\omega_w} = 0 = R_{\alpha_w}$ implies $\omega_w = \alpha_w$; hence w is fully commutative.

(ii) $\Rightarrow$ (iv). By Proposition 3.2, given a reduced decomposition $\mathbf{i}$ and its pseudo-line arrangement Γ, the quadratic terms in $P_{\mathbf{j}}(x_1, ..., x_n)$ are in one-to-one correspondence with the efficient pseudo-lines in Γ. ∎

Figure 6.2 shows the poset $\mathscr{C}(w_0)$ for $w_0 = 4\,3\,2\,1 \in S_4$. Each commutation class is represented by its smallest reduced decomposition for the lexicographic order. Each 3-move is represented by an arrow going from the smaller class to the bigger class for the partial order $<_c$. The extremal classes α_{w_0} and w_{ω_w} consist of two reduced decompositions. The classes in

$$\begin{tikzcd}
 & (1,2,1,3,2,1) & \\
(1,2,3,2,1,2) \arrow[ur] & & (2,1,2,3,2,1) \arrow[ul] \\
(1,3,2,1,3,2) \arrow[u] & & (2,1,3,2,1,3) \arrow[u] \\
(3,2,1,2,3,2) \arrow[u] & & (2,3,2,1,2,3) \arrow[u] \\
 & (3,2,1,3,2,3) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

FIGURE 6.2

the middle row have four reduced decompositions each. The four remaining classes are of cardinality 1 (see Fig. 1 in [BZ, Sect.9]).

When $w_0 = 5\ 4\ 3\ 2\ 1 \in S_5$, the set $\mathcal{R}(w_0)$ has 768 elements and $\mathcal{C}(w_0)$ has 62 elements (see [Kn, p. 35], where A_n denotes the number of reduced decompositions of $w_0 \in S_n$, and B_n the number of commutation classes). Robert Bédard showed us (private communication) that the poset $\mathcal{C}(w_0)$ can be drawn on the surface of a 2-dimensional sphere S^2. In other words, its geometric realization is homeomorphic to S^2.

It would be interesting to determine the topology of the poset $\mathcal{C}(w_0)$ when w_0 is the longest element of S_n for $n \geqslant 6$.

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